How Software can Revolutionize the Way We Teach Graphing a Function

Vladimir Nodelman
Email: nodelman@hit.ac.il, vladinod@gmail.com

Department of Computer Sciences,
Holon Institute of Technology
ISRAEL

Abstract

One of the typical tasks in the study of functions is sketching of their graphs. The acquired skills, as well as the ability to read graphs, are notably useful in the later life [2], [3].
The software used in teaching mathematics automatically builds ready-made graphs. While studying sketching these graphs are unacceptable.
Traditionally, sketching done by hand on paper. Nowadays (rarely) this can be done on a computer monitor as a freehand drawing. The resulting curves are approximate and rough.
This paper illustrates the method of sketching graphs with intensive use of software.
In general, the types of student's activities do not depend on the specificity of functions. They explore a common model with the help of author's non-profit software "VisuMatica". This model includes a function $y = f(x)$ with a hidden graph and the ability to visualize its characteristic properties:

- *Intervals of sign constancy,*
- *Discontinuities,*
- *Local max/min,*
- *Supremum/infimum,*
- *Critical points,*
- *Asymptotes,*
- *Domain/Range,*
- *Convexity and inflection points.*

Using the approaches demonstrated in this paper, student builds a sketch of the graph (one or more of its branches), by setting control points. The program connects them by smooth curves. These curves can be edited by dragging, exact locating, adding and removing control points.
The result checked by unhiding the graph and comparing it with the constructed sketch.
The model is universal: to build a sketch of a graph of another function, it is enough to redefine the expression of $f(x)$ and the whole show (except of the old sketch) rebuilds automatically.

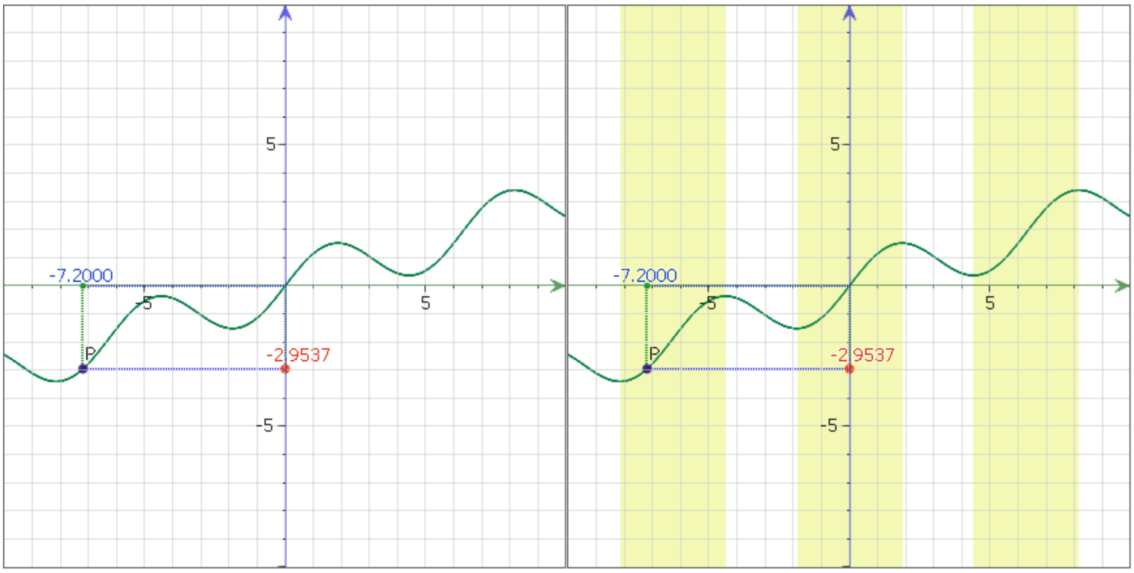
1. Software in function's studies

The characteristic properties studied from the very beginning of introduction of the concept of function. A suitable software can help students to grasp their meaning. Unfortunately, the known software sometimes not only do not support the necessary educational activities, but also incorrectly represent the graph of the function itself, in particular, its continuity and isolated points.

Let us see how *VisuMatica* can help in this matter.

1.1. Increasing, Decreasing and Monotonic functions

Not formal interpretation of function increasing is *the increasing of the value of f(x), when the argument x grows*. Naturally, x grows from left to right along the x-axis. The proper model it extremely simple (Fig.1 a): the graph of some function, $f_1(x)$, say, $y = \sin(x) + ax$ ($a = 0.3$) and a point on it, which defined as P(b, $f_1(b)$). The point P projected onto coordinate axes. The green point on the x-axis presents abscissa ($x = b$) and the red point on the y-axis – ordinate ($y = f_1(b)$) of point P.



a) b)

Figure 1

During the animation[1] of parameter b, point P slides across the graph from left to right, and students observe the behavior of the function by watching the movement of the red point and the changing number (red) - the value of $f_1(b)$. They discover points of increasing and decreasing in accordance to the movement of red point. After changing the value of parameter a to $a = 1$ (a = -1) and repeating the animation of parameter b students notice that the new function increase (decrease) in all points of its domain. Set the value of a back to 0.3. In this case, as we saw, the points where function increase/decrease are not isolated, but fill in certain intervals on the abscissa axis.

Trying to describe in words the observed situation of increasing function, we bring students to the following formulation: "a function increases if its value at the "nearest-next" point is greater than at the previous one".

VisuMatica includes a variable TINY as a representative of the concept of "tiny" positive number. This helps to formulate the last "definition" in the form of the following inequality $f_1(x + \text{TINY}) >$

[1] In *VisuMatica*, when any parameter is animated, its value cyclically increases within the corresponding interval.

$f_1(x)$. After adding this inequality our model (Fig.1 b) presents the increasing intervals as yellow strips[2]. Check the correctness of the show by animation of the *b* parameter.

- Check what happens when parameter $a = 0.99, 1, -1$. Explain your observations.
- Redefine the inequality by $f_1(x + \text{TINY}) < f_1(x)$. Play with values of *a*. Explain your observations.

Acquaintance with the concept of a derivative allows seeing in these inequalities prototypes of conditions for the sign constancy of the derivative. Really, in case of increasing:

$$f_1(x+\text{TINY}) > f_1(x) \Leftrightarrow f_1(x+\text{TINY}) - f_1(x) > 0 \Leftrightarrow \frac{f_1(x+\text{TINY}) - f_1(x)}{\text{TINY}} > 0 \ldots f_1(x)' > 0 .$$

The awareness of this fact makes it easier for students, based on their experience, to understand that if *function f*(*x*) *is differentiable on interval* (*m*, *n*), then there are the following *dependencies between the character of monotonicity and the sign of the derivative on the interval* (*m*, *n*). In brief, we demonstrate this in Fig.2.

$$\boxed{\begin{array}{c} f'(x) > 0 \Rightarrow f(x)\uparrow \\ f(x)\uparrow \Rightarrow f'(x) \ge 0 \\ f'(x) \ge 0 \Leftrightarrow f(x)\nearrow \end{array}} \quad \boxed{f'(x) = 0 \Leftrightarrow f(x) = const} \quad \boxed{\begin{array}{c} f'(x) < 0 \Rightarrow f(x)\downarrow \\ f(x)\downarrow \Rightarrow f'(x) \le 0 \\ f'(x) \le 0 \Leftrightarrow f(x)\searrow \end{array}}$$

Figure 2. Increasing, stability and decreasing of the function *f*(*x*).

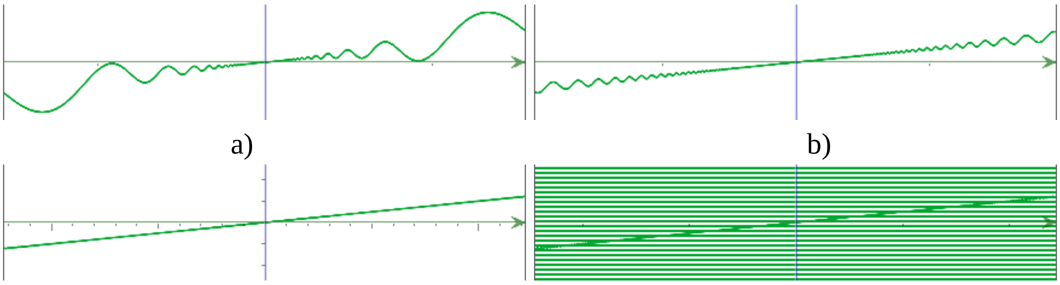
The geometric meaning of these statements gets clear after adding tangent $y = f_1'(b)(x- b) + f_1(b)$ to the model and varying the *b* parameter. This way, students internalize the relationship between the increasing, decreasing and "stability" of the function and the slope of the tangent.
A discussion of the meaning of these statements and the nuances of the dependencies between them becomes most productive when considering different examples with the help of this model.
As always, counterexamples help deeper understanding of the studied subject.
Particular attention should be paid to the fact of definite sign of the derivative over the entire interval (*m*, *n*) in all the above statements.
As a counterexample, we consider the following differentiable on **R** function

$$f(x) = \begin{cases} \dfrac{x}{10} + x^2 \sin\dfrac{1}{x}, \text{ if } x \ne 0 \\ 0, \text{ if } x = 0 \end{cases} \quad \text{(Fig.3 a).}$$

The function's derivative at point $x = 0$ is positive $f'(0) = \lim\limits_{d\to 0} \dfrac{\dfrac{d}{10} + d^2 \sin\dfrac{1}{d}}{d} = \dfrac{1}{10} > 0$, but the damping down fluctuations of the graph near zero arouse suspicion. Let us zoom in closer to check if the function really increases at point $x = 0$. After few clicks on "*Zoom In*" button  of the taskbar our graph becomes a straight line (Fig.3 b, c), which is surely increasing on **R**. One can even specify its

[2] Abscissas of yellow points satisfy inequality.

slope: it is 0.1. Strange! Let us try to see if there are critical points in this interval. After all, the only intervals of monotonicity are those that lie between neighboring critical points of function $f(x)$. In result of enabling option "*show critical points*",[3] our view becomes filled with vertical dotted lines (Fig.3 d). These intervals are contracting to a point as they approach the point 0. There is no such interval $(-b, b)$ with $b > 0$, which includes the zero point. Thus, function $f(x)$ not increase nor decrease at point $x = 0$ and not monotonic on any interval, that includes $x = 0$, despite the fact that the derivative at zero exists and is positive.

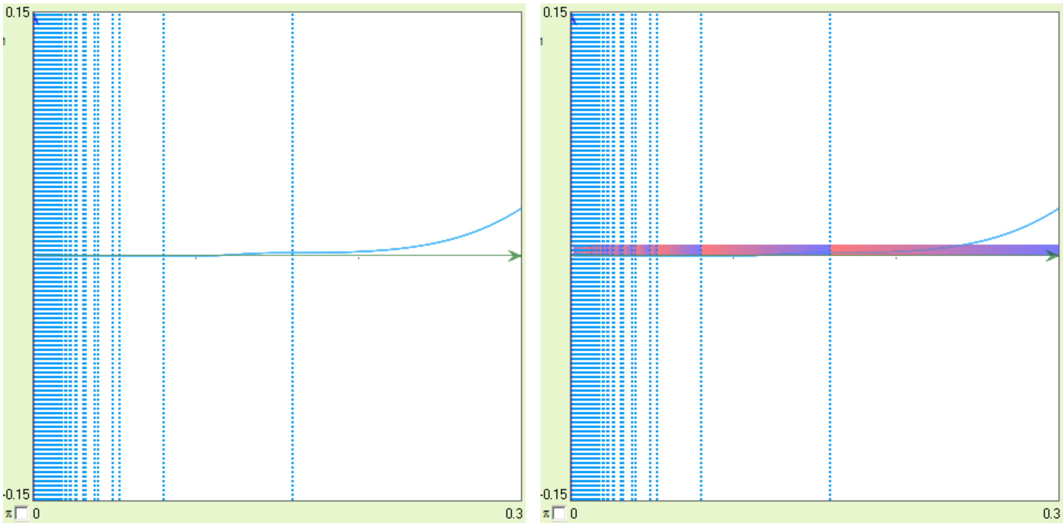


a) b) c) d)

Figure 3. Visible intervals:
a) [-0.15625, 0.15625] , b) [-0.1953125, 0.1953125], c), d) [-0.001220703125, 0.001220703125]

Consider following example of continuous, increasing on a finite interval $(0, a)$ function with a non-negative derivative.

$$f(x) = \begin{cases} e^{\sin\frac{1}{x} - \frac{1}{x}}, \text{ if } x > 0 \\ 0, \text{ if } x = 0 \end{cases}, \qquad f'(x) = e^{\sin\frac{1}{x} - \frac{1}{x}} \left(1 - \cos\frac{1}{x}\right)\frac{1}{x^2} \geq 0.$$



a) b)

Figure 4

[3] To show critical points of a function - point with mouse to the legend of this function, press right mouse button and select the proper submenu in the appeared pop-up menu. Critical points presented by dotted vertical lines that consist of all the points with the same abscissa – the critical point.

(Fig.4 a) presents graph of this function. The graph increases so slowly that it is difficult to see its behavior. Simple zoom does not help[4]. Enabled show of the critical points makes things clear: the derivative equals to zero infinitely many times on (0, a). Really,

$$f\,'(x) = 0, \text{ when } x = \frac{1}{2\pi n} \quad (n = 1, 2, 3, \ldots).$$

Abscissas of the vertical lines present these zeros of the derivative.
Enabling the show of "*critical points and monotonicity* (*intervals*)" (Fig.4 b), we find that function increases between the neighboring critical points[5], thereby confirming that $f'(x) \geq 0$. It is worth asking students to explain this indirectly, observing the function behavior.

All the considered statements (Fig.2) were valid subject to the <u>requirement of differentiability of function $f(x)$ on the interval (a, b).</u>Can a function be monotonic if it is non-differentiable at some points?
To check it we take a differentiable function $y = x$ and turn it into a non-differentiable at point x=0 by taking modulo: $y = |x|$. This function is not monotonic. To "correct" the situation add to it the function $y = k\,x$, and let $k \geq 1$. Voila! For $k = 1$ our new function $y = |x| + k\,x$ has become monotonic, although weakly. Replacing the value of parameter k with any number bigger than 1 we obtain a *strictly monotonic function over the entire domain, although it is not differentiable at the point 0* (Fig.5 a-c).

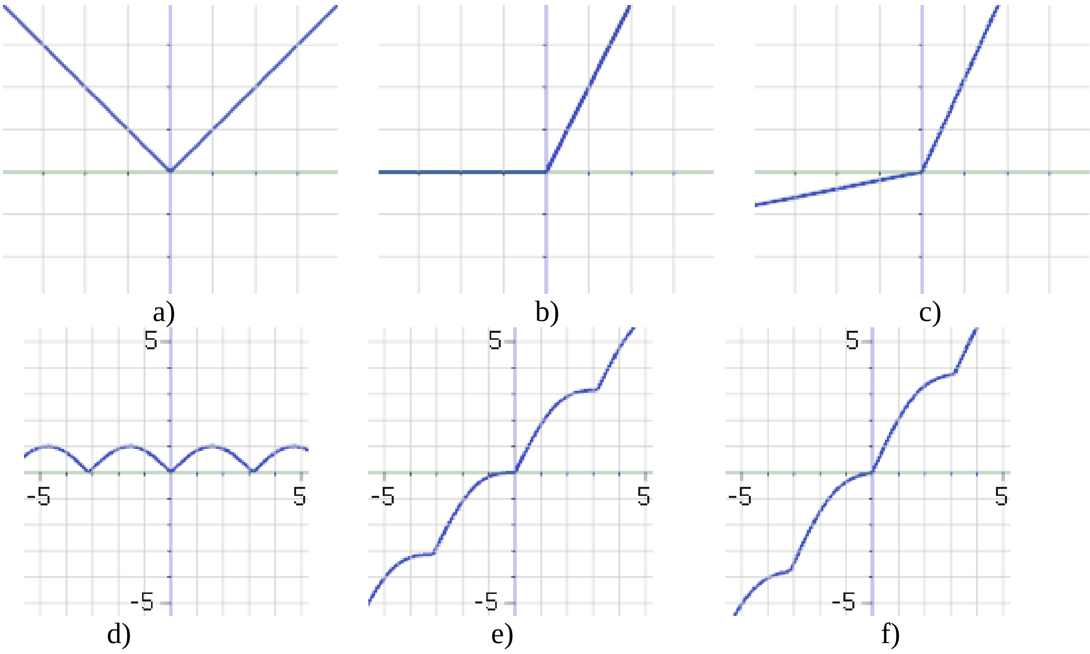


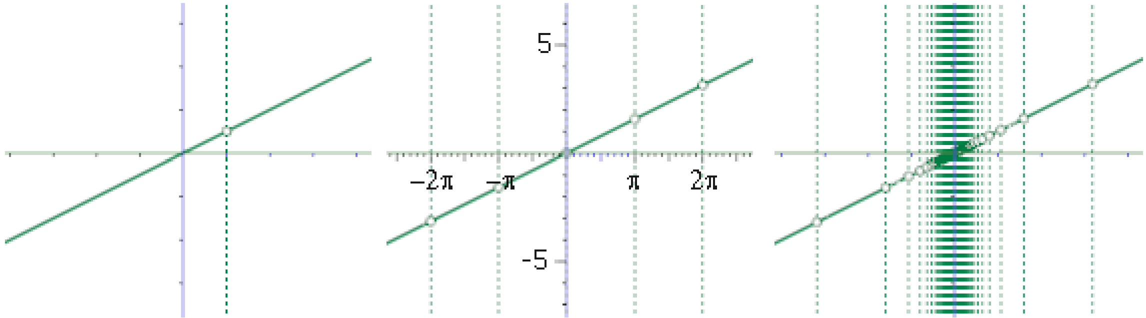
Figure 5. $y = |x|$ (a), $y = |x|+k\,x$, $k = 1$ (b), $y = |x|+k\,x$, $k = 1.2$ (c)
$y = |sin(x)|$ (d), $y = |sin(x)|+k\,x$, $k = 1$ (e), $y = |sin(x)|+k\,x$, $k = 1.2$ (f)

[4] To grasp the idea of behavior change the visible y-axis interval to (-0.015, 0.015) and (-0.000015, 0.000015).

[5] The change from red to blue on an interval means that the function on it increases, and vice versa: the transition from blue to red means an interval of decreasing function.

The same steps related to the differentiable on R function $y = \sin(x)$, lead us to the function $y = |\sin(x)| + k\,x$ - an example of a strictly monotonic continuous function with an infinite number of points at which the derivative is absent (Fig.5 d-f).

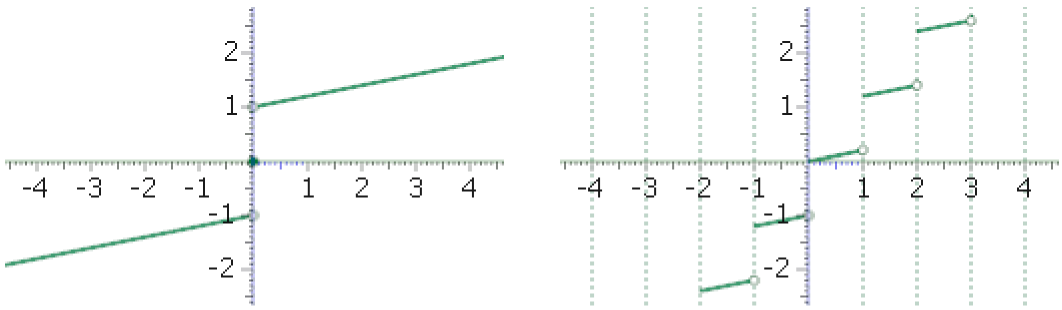
- Give examples of strictly decreasing and weakly decreasing continuous functions with a finite and infinite number of points with a violation of differentiability and check these examples using *VisuMatica*.
- Are the discontinuous - and therefore non-differentiable at the points of discontinuity - functions shown in Fig.6, monotonic? Define them and check using *VisuMatica*.
- Give examples of strictly decreasing functions with a finite and infinite number of points of discontinuity, and check these examples using *VisuMatica*.



a) b) c)

Figure 6

These were examples of *monotonic functions with removable discontinuities.*



a) b)

Figure 7

- Using the familiar examples of weakly monotonic functions containing jump discontinuities: y = sign (x) and $y = [x]$, define examples of strictly monotonic functions with a finite and infinite number of jump discontinuities (Fig.7).
- Prove that there is no monotonic function containing an essential discontinuity (discontinuity of the second kind)

1.2. Local extrema

We say that function f has a *strict local maximum* (*strict local minimum*) at point a if
$(\exists b > 0)(\forall x \in [a-b, a+b])\ f(x) < f(a) \quad (f(x) > f(a))$

We say that function f has a *non-strict local maximum* (*non-strict local minimum*) at point a if $(\exists b > 0)(\forall x \in [a-b, a+b])\ f(x) \le f(a) \quad (f(x) \ge f(a))$

All these definitions deal with four objects:

- the function $f(x)$,
- the point $(a, f(a))$,
- the inequality between $f(x)$ and $f(a)$,
- the b neighborhood of a, which can also be expressed as inequality $|x - a| \le b$.

Consider the following model, based on the graph of function $f(x) = 0.5(x-2)^2 - 5$ (Fig.8).

The definitions of *local extrema* are implemented here by point M(a, $f(a)$), by solution of the system of inequalities $|x - a| \le b$; $f(x) > f(a)$ (in green), and by solution of the system of inequalities $|x - a| \le b$; $f(x) < f(a)$ (in cream).

- Does the function have a local extremum at the point $x = 2$, $x = 2.4$, $x = 4$? How to "verify" the correctness of the answer via changing the value of parameter b in the model?
- Is it possible to give a concrete answer to the presence of a local extremum at a point using the model immediately, without trying to select the appropriate value of the parameter b? If the answer is "YES", then explain on what basis the conclusion will be that the point is a local extremum, and in which it is not.
- Select the initial function $f_1(x)$ and redefine it as $y=0.5[x] + \sin(x)$ (Fig.9). Describe all the points of local maximum and local minimum of the function. Use model to check your answers.

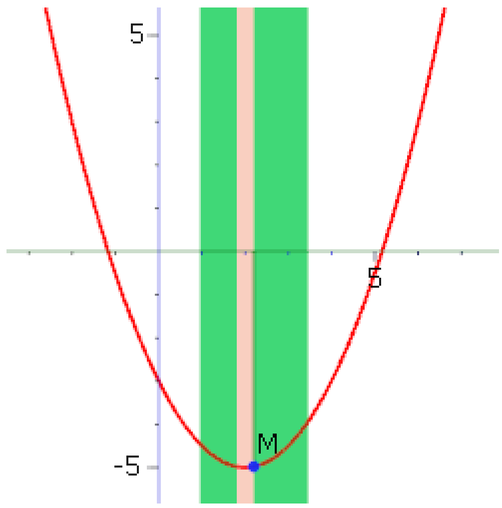


Figure 8. $a = 2.2$, $b = 1.25$

A sufficient condition of extremum:

Let exists such $b > 0$ that $f(x)$ is continuous on $|x - a| < b$, exists $f'(x)$ on $0 < |x - a| < b$ and $f'(x_0)$ does not exist or equal 0. If $f'(x_1) \cdot f'(x_2) < 0$ for every x_1, x_2: $x_0 - b < x_1 < x_0 < x_2 < x_0 + b$ then x_0 is a point of local extrema.

- Express this condition in terms of our model.
- Prove that the converse statement is not true. Give a counterexample to the claim that if a differentiable function has a strict maximum at point x_0, then there is a neighborhood of this point, where the function increases to the left of x_0 and decreases to the right. Check it with *VisuMatica*.

<u>A necessary condition of extremum</u>:
Let x_0 be a local extrema of function f and there exists $f'(x_0)$ then $f'(x_0) = 0$.

- Give a counterexample to disprove the opposite statement, and illustrate it VisuMatica. Have we seen a similar case before?

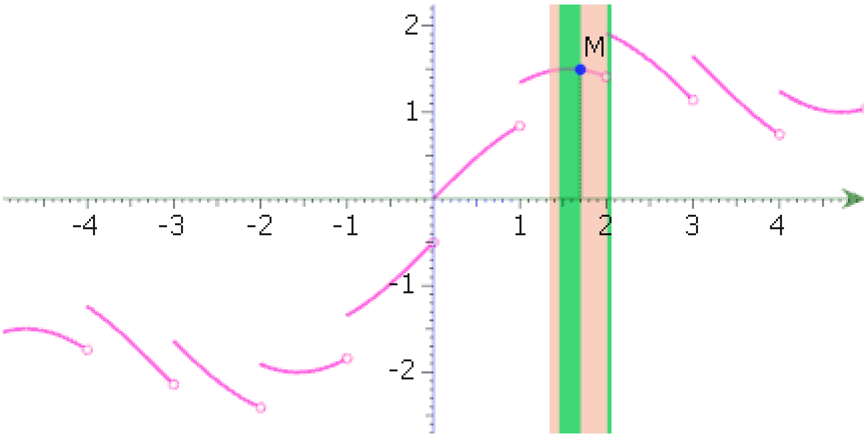


Figure 9. *a*=1.7, *b*=0.36

In conclusion, we note that *VisuMatica* allows you to emphasize automatically the points of local extrema. To do this, position the mouse pointer to the legend of the function, press the right mouse button and select "*show local max/min*".
Fig.10 shows the result of such a choice with respect to the function

$$y=\begin{cases}1+x^2\left(2+\sin\dfrac{1}{x}\right), & \text{if } x\neq 0\\ 1, & \text{if } x=0\end{cases}$$

. The points of the graph corresponding to the local maximum are in light blue, and to the local minimum are in light red.

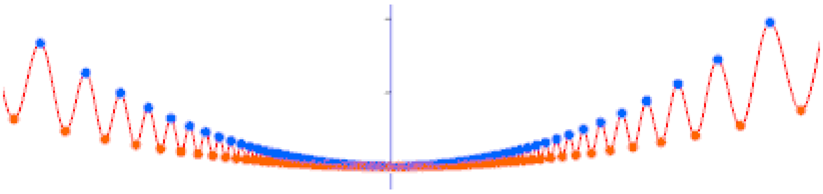


Figure 10

- How to recognize intervals of monotonicity of a function with enabled option "*show local max/min*"? Is it always possible?

Note to students that the definition of extrema points does not include the concept of derivative.

- Consider examples in Fig.5. Guess their local extrema and check your answers with *VisuMatica*. Is there a derivative at the points found?
- Can a function have both a maximum and a minimum at the same point? If your answer is "Yes", then give an example. How one can see it in the considered models with enabled option "*show local max/min*"?

The counterexample in section 3 of [1] demonstrates a very special case of a *continuous function*, that is *nowhere differentiable, nowhere monotonic*, and "*everywhere" has extrema points*.

VisuMatica allows also displaying the function's infimum(s) and supremum(s): just select "*show sup/inf*" after pressing the right mouse button on function's legend. Corresponding points if any will appear on the graph as large blue and red circles. Fig.11 presents visual solution of the problem of finding $\min_a \sup_{|x|\le 1} |x^2 + a|$ by means of graph of function $y = |x^2 + a|$ *if* $|x| \le 1$ with enabled option "*show sup/inf*".

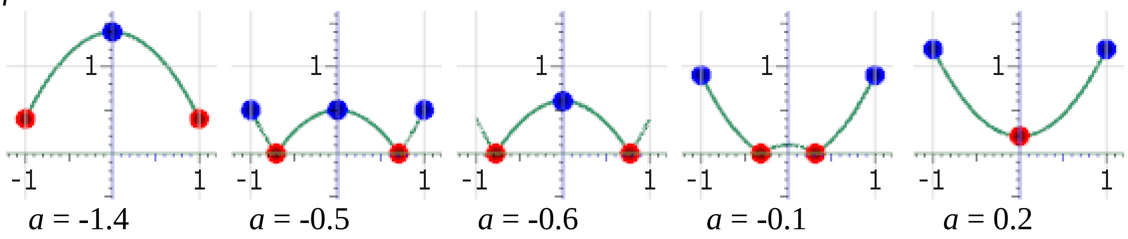


Figure 11

1.3. Convexity of function. Inflection points

Monotonicity and extrema are not the only characteristics of the function behavior. Both graphs in Fig.12 a) represent increasing, and in Fig.12 b) - decreasing functions, but all the four graphs have a common feature – they lie on a one side from the tangent to any point on graph in a neighborhood of this point. Even the not monotonic functions in Fig.12 c) have the same feature.

- When this property fails? Provide a counterexample.

At the same time green curves, as well as orange curves are somehow similar in shape. The curves in different colors "bend" in different directions, while ones with the same color "bend" in the same direction.

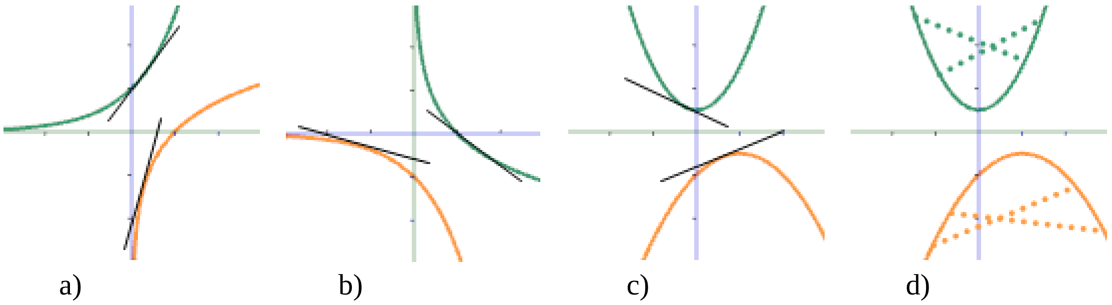


Figure 12

Fig.12 d) clarifies our observations. The green segment connecting any two points on the green graph lies above the green curve between these points, while the orange segment lies under the orange curve between the segment's endpoints.

To make general conclusions consider the following model, based on the graph of function $f_1(x) = 0.1(x+2)^2 - 3$ (Fig.13). Here the endpoints of segment P_1P_2 controlled by x_1 and x_2-points of their projection onto x-axis. Points x_1 and x_2 are movable manually along the x-axis. Point x_0 defined as $x_0 = (1-k)x_1 + kx_2$, where parameter $k \in [0, 1]$. Point P_0 belongs to the segment P_1P_2 and point $\boldsymbol{f_0}$ belongs to the graph of function $f_1(x)$. Both of them have abscissa x_0. The model also includes a red segment-arrow P_0f_0.

By changing with slider the value of t pay attention that:

- ✓ The graph on interval $[x_1, x_2]$, highlighted in magenta, lies under the segment P_1P_2.
- ✓ The red arrow always remains directed downward.

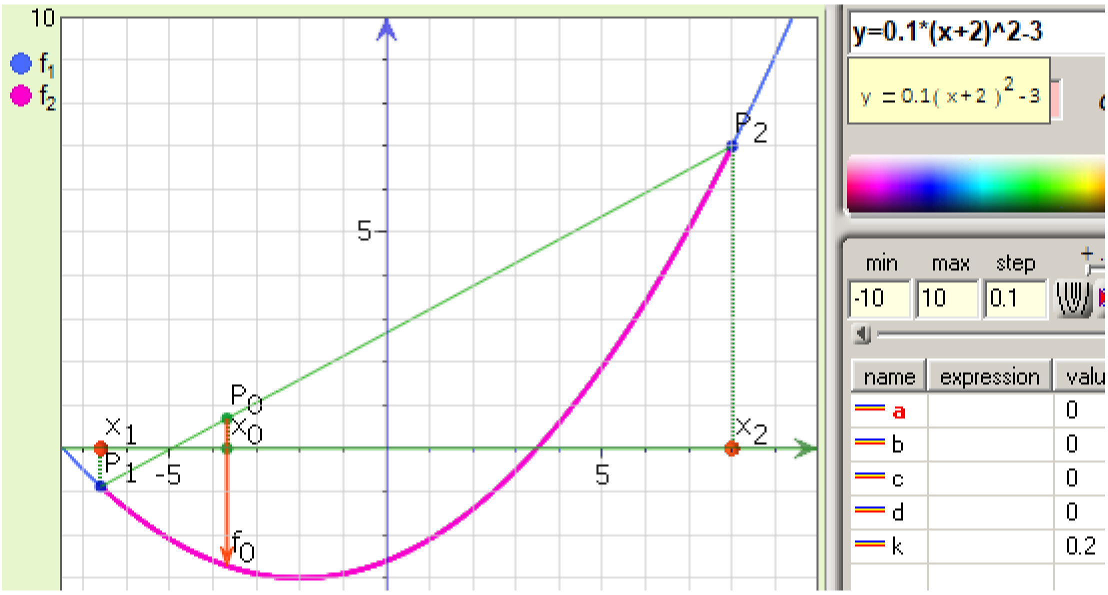


Figure 13

Both of these properties preserved when changing the values of x_1 and x_2 by moving these points along the x-axis.

After note that the ordinates of points P_1, P_2, P_0, $\boldsymbol{f_0}$ respectively $f(x_1)$, $f(x_2)$, (1 - k) $f(x_1) + k\, f(x_2)$, $f((1 - k)\, x_1 + k\, x_2)$ stunts are ready for the following formal definition:

We call function f <u>convex *downward (convex) on X*</u> if

$$\forall x_1, x_2 \in X, \forall k \in [0,1] \text{ holds } f((1-k)x_1 + kx_2) \le (1-k)f(x_1) + kf(x_2)$$

- What is the meaning of the right and left side of this inequality?
- Provide examples of convex functions and check your answers with *VisuMatica*.

Redefine $f(x_1)$ to $-f(x_1)$. In our case it becomes $y = -0.1(x+2)^2 + 3$.

The graph turned upside down, and with it the red arrow. Now:

- ✓ The graph on interval [x_1, x_2] lies above the segment P_1P_2.
- ✓ The red arrow directed upward.

After some play with parameter t and points x_1, x_2 it is easy to grasp the following definition:

We call function *f* <u>*convex upward (concave) on X*</u> if

$$\forall x_1, x_2 \in X, \forall k \in [0,1] \text{ holds } f((1-k)x_1 + kx_2) \geq (1-k)f(x_1) + kf(x_2)$$

- Provide examples of concave functions and check your answers with *VisuMatica*.
- Provide counterexamples of concave/convex functions and check your answers with *VisuMatica*. How the set *X* affects the presence of convexity of a function?
- What color have the convex upward/convex downward function in Fig.12?

Remind our observation about location of the graph of a convex/concave function with respect to the tangent (Fig.12). To verify it, we introduce:

- a new variable *derivative* - the derivative of function $f(x)$ at x_0 as *derivative*: = f_1 ($x_0.x$)',
- a red tangent to the graph at point x_0 as y = *derivative* * (x-$f_0.x$) + $f_0.y$,
- a yellow half-plane that lie above the tangent by the inequality y > *derivative* * (x-$f_0.x$) + $f_0.y$.

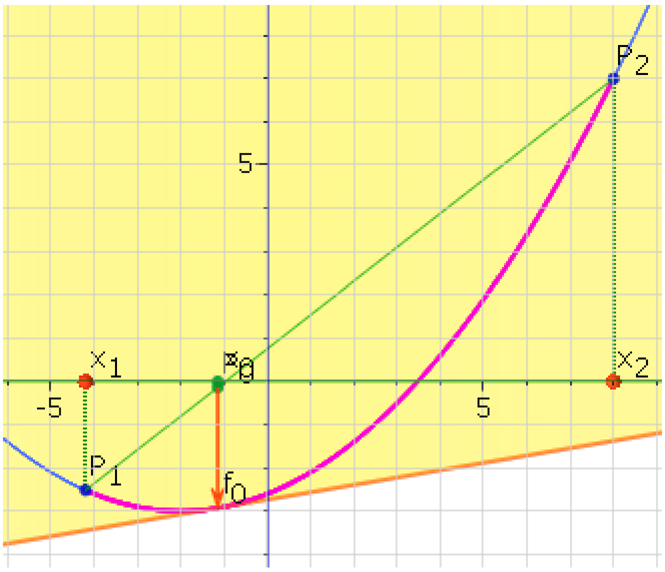


Figure 14

Fig.14 shows the updated model. Select the k variable (click on it) and redefine its limits from 0, 1 to *xMin, xMax* correspondingly.

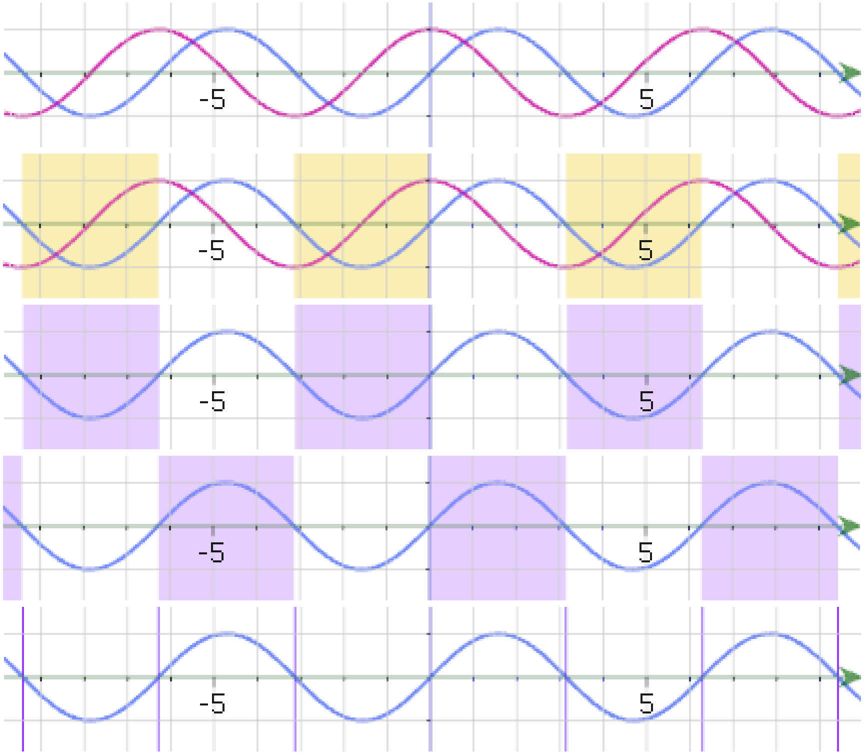
By assigning all possible values from the visible segment of the x-axis to the variable k, we make sure that the graph is always completely located inside the colored half-plane, which means that it lies above the tangent.

Similar activities one can organize in case of a concave function.

Pay attention to the value of parameter *derivative* when grows the value of parameter k (the point x_0 moves from left to right). Try to grasp the link between the *convex/concave feature of a function* and *monotonicity of its derivative*. Check conclusions by adding the graph of the second derivative y = $f_1(x)$''.

Start *VisuMatica* and enter some counterexample – a function that is not convex overall domain, say, $y = \sin(x)$ (in blue). Of course, we can add a graph of the derivative $y=f_1(x)'$ (in magenta) and consider the intervals at which the derivative increases/decreases. They will be the intervals of convexity/concavity of the original function $f_1(x)$ (Fig.15 a). However, this observation requires some effort.

Instead, let us try to provide the condition for increasing the derivative in a "primitive" form $f_1(x+\text{TINY})' > f_1(x)'$. The picture has become much more expressive (Fig.15 b). Reminding, that at intervals of increase the sign of the function's derivative is greater than zero, we replace the previous "primitive" condition with the correct one with the second derivative: $f_1(x)'' > 0$ (Fig.15 c).



a)
b)
c)
d)
e)

Figure 15

By changing the sign of inequality, we immediately obtain intervals of concavity (Fig.15 d).

Where the function is neither convex nor concave? - When the second derivative equals to zero. We get the vertical lines after adding equation $f_1(x)'' = 0$ (Fig.15 e). At roots of this equation, the second derivative changes its sign. From concave, the function becomes convex and vice versa.

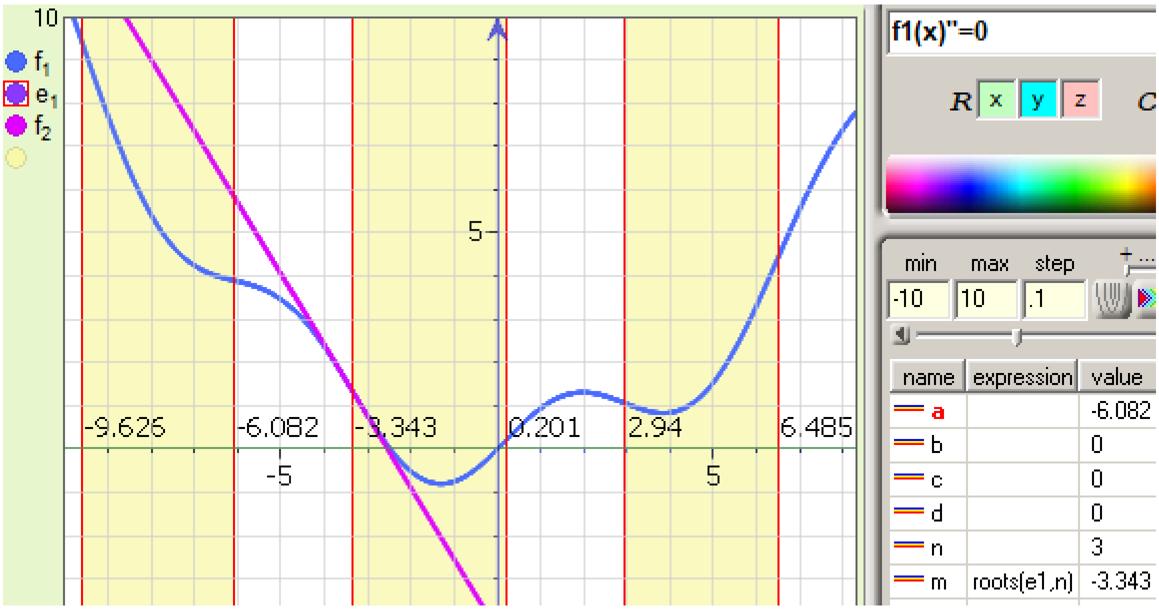


Figure 16

Fig.16 presents a model, which shows the relationship between convexity, concavity and inflection points of a function with values of the second. It includes an inequality $f_1(x)'' > 0$, which highlights in yellow the intervals of function concavity, and equation $f_1(x)'' = 0$ (e_1) with roots presented by vertical lines and their values. A tangent to the graph $y = f_1(m)'(x - m) + f_1(m)$ shown in magenta. The m parameter here presents values of roots of the second derivative. To receive these values it defined as m: = *roots* (e_1, n), where e_1 is the legend name of equation $f_1(x)'' = 0$, and n is the index number of the specific root.
It is easy to see that the graph of $f_1(x)$ lies on both sides of each tangent to graph at these roots by assigning integer values to parameter n.

Such points called *inflection points*. Namely,

Let the function f be differentiable in the punctured neighborhood $[x_0 - d, x_0) \cup (x_0, x_0 + d]$, where $d > 0$. If on $[x_0 - d)$ the function is convex/concave and on $(x_0 + d]$ it is concave /convex, then the point $(x_0, f(x_0))$ of the function graph is called its *inflection point*.

- Give an example of such a function f for which x_0 is not an inflection point, although $f(x_0)'' = 0$, and check your answer with *VisuMatica*.
- Provide an example of a function that is not differentiable in its inflection point and check your answer with *VisuMatica*.

2. Sketching the function graph

In the previous section, we limited ourselves to considering the capabilities of the software when studying some of the characteristics of a function.
The sketching of a function graph is based on the results of studying its various properties.
A typical sequence of steps when sketching a function is as follows:

- Ascertainment of the domain, evenness, oddness and periodicity of the function,

- The search for discontinuity points of the function and their classification, finding the vertical asymptotes of the graph,
- Finding the intersection points of the graph with coordinate axes,
- The search for critical points of the function, the allocation of points of extrema, the determination of intervals of monotonicity,
- The search for inflection points and function values at these points, establishing intervals of convexity,
- Finding inclined or horizontal asymptotes of the graph functions.

Fig.17 a) presents the skething model, which initially includes five objects:

- (f_1) Graph of function y = 4sin(x-1)/(x+2) +1 in purple. **Hidden**.
- (e_1) Solution of equation $f_1(x)$" = 0 as blue marks on the x-axis: the inflection points.
- (f_2) Graph of the derivative $f_1(x)$' in green. **Hidden.**
- (f_3) Graph of the second derivative $f_1(x)$" in blue. **Hidden.**
- A purple point (0, $f_1(0)$) on the y-axis: the intersection point of the graph with the y-axis.

The user interface[6] (Fig.18) allows visualization of the characterization features of $f_1(x)$.
All the correspondent drawings: rectangular regions, vertical lines, arrows, isolated points and asymptotes automatically painted in the same color as $f_1(x)$ - in this example: purple.

- Intervals of sign constancy emphasized by light purple rectangles. Their sides located on the x-axis present these intervals[7]. These rectangles are oriented in the positive (upward) or negative (downward) direction according to the sign of the function on them. Naturally, all points of the graph lie inside these rectangular areas.

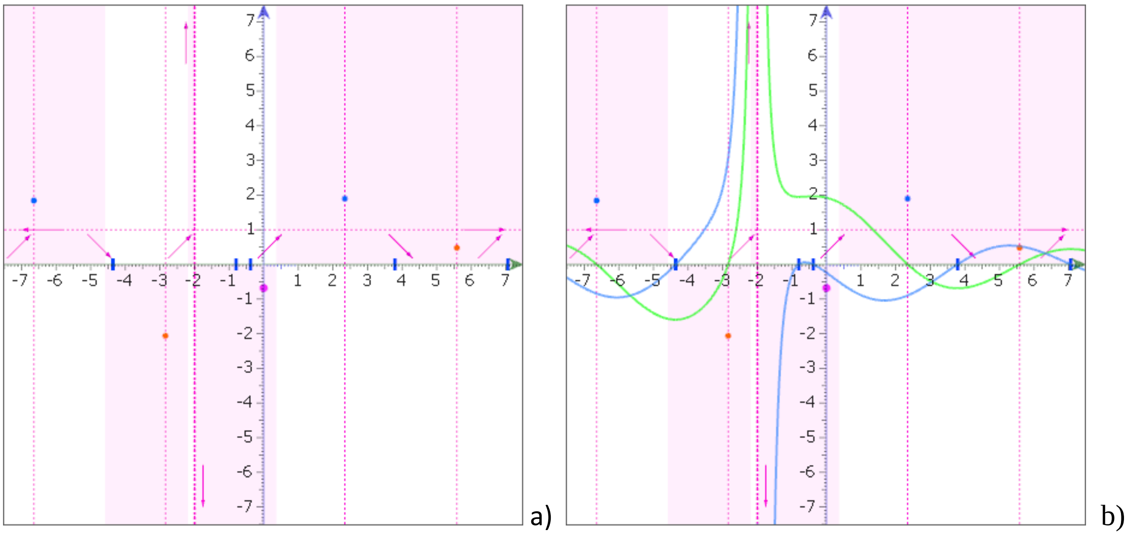


Figure 17

[6] This pop up menu becomes visible when user presses right mouse button on the icon of any function $f_i(x)$ in the legend.

[7] The ends of these sides-intervals are the function's zeros.

- The wide vertical dotted line at x = -2 with arrows[8] presents the vertical asymptote.

  The upward arrow on the left of vertical asymptote hints that the graph increases without bounds while approaching to the asymptote when $x \to -2^{-}$. The downward arrow on the right of this asymptote tips-off that the graph decreases without bounds while approaching to the asymptote when $x \to -2^{+}$.

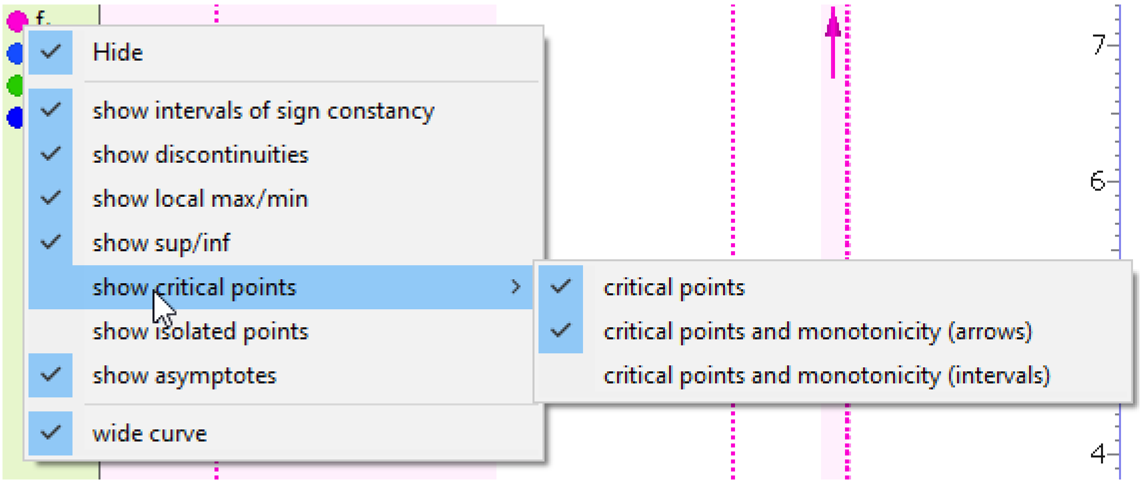


Figure 18

The wide horizontal dotted line at y = 1 presents the vertical asymptote.

  The two arrows on this line directed left and right prompt that graph approaches this asymptote as x tends to +∞ and -∞.

- The vertical dashed line at x = -2 shows the discontinuity point. It coincides with vertical asymptote.
- The local maxima presented by the blue points and minima – by the red ones. There are neither suprema not infima (otherwise, big blue/red points would display them).
- The critical points, where the derivative equals to zero or does not exist, displayed by vertical dotted lines (one of them, where the derivative does not exist, coincides with the vertical asymptote). The arrows in the middle of the intervals between the critical points (intervals of monotonicity) according to the signs of the derivative indicate the directions of function's increasing / decreasing. The arrow not drawn if the function is constant on this interval or interval is too narrow in current view.

In case of difficulty, one can unhide graphs of the 1st and 2nd derivatives (Fig. 17 b).

In this regard, it is not too late to note an important specialty of the derivative function*: the domain of the derivative f ' is the set of all points in the domain of f at which f is differentiable, i.e. the domain of f ' is a subset of the domain of f.*

This obviously follows from the definition of the derivative, which depends on f(x).

[8] Arrows on asymptotes appear when the graph of function is hidden.

For instance, considering the derivative of the logarithmic function, students should pay attention to the fact that it is $y = 1/x$, <u>when $x > 0$</u>. Fig.19 shows graphs of following three functions:

$f_1(x)$: $y = \ln(x)'$ in red,

$f_2(x)$: $y = (\sin(x) + \ln(x))'$ in dark green, and

$f_3(x)$: $y=\cos(x) + 1/x + 0.1$ in light green.

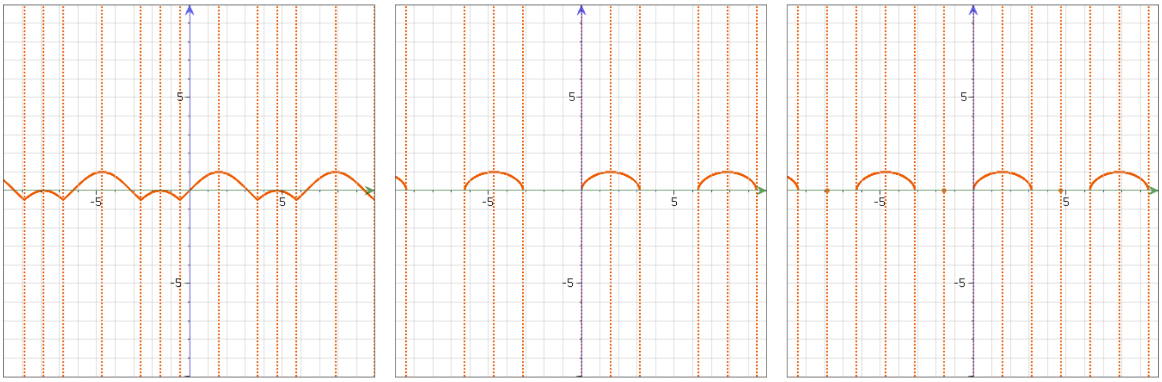
VisuMatica distinguishes between the derivative of a function (f_1 and f_2) and f_3-the function-result of taking the derivative (we added the term 0.1 to make the right branch of the graph of function f_3 and the entire graph of the function f_2 distinguishable).

Figure 19

This dependence between domains of the derivative and the function itself reflected in the exhibition of various characteristics of the derivative function. Fig.19, for instance, shows the points of local maximum and minimum of the derivative f_2, which are exclusively within its domain.

A more interesting case is the option to "*show isolated points*" of function (Fig.18). By default, these points not displayed when displaying the function graph. A thoughtful user may suspect their presence and request to show them.

Consider, for example, the graph of the function $y = |\sin(x) + 0.5| - 0.5$. With enabled show of critical points, it looks rather understandable (Fig.20 a). Now let's replace this function with the square root of its expression $y = \text{sqrt}(|\sin(x) + 0.5| - 0.5)$ (Fig.20 b). It is quite obvious that even though the sections of the graph with negative values of the original function have reasonably disappeared, at the same time the points of touch of the x-axis from below, which should still belong to the new graph, have disappeared. Enabling option "*show isolated points*", we get these points on the screen[9] (Fig.20 c).



a) b) c)

Figure 20

Based on the analysis of the behavior of the function and its derivative, let us try to sketch its graph. For this purpose, we will use the "*Freehand cu*rve" mechanism, and launch it by clicking the toolbar button .

From now on, each click of the left mouse button adds a control point of the future smooth curve. This work completed by clicking the right button, which adds the last point, resulting construction and show of the final curve, which approximates the control points.

Consider the application of this mechanism using the example of the model presented in Fig.17.

It is clear that graph in the visible interval has two branches on both sides of the vertical asymptote x = -2. When starting sketching the left branch we take into account both asymptotes, especially the vertical one: the curve goes to + ∝ when *x* approaches to -2 from left, according to the direction of left purple arrow next to the vertical asymptote in Fig.17 a).

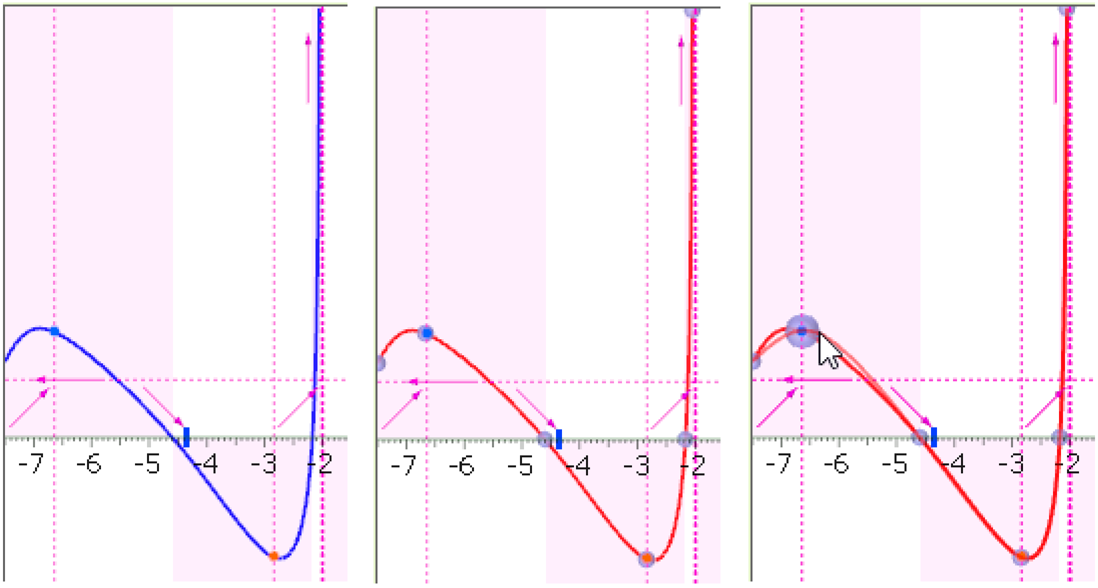
In addition, the graph for sure passes through the common vertices on the *x*-axis of the light-purple zones of constant sign, and through two more points: the blue point of the local maximum and the red point of the local minimum. Well, the first should be some point at the left border of the visible domain. It should be below the local maximum, because there the function increases, judging by the direction of the left oblique arrow.

We start the curve construction mode, and by clicking on these points, we get the curve shown in Fig.21 a).

Unpleasant result: we tried to click as accurately as possible on the previously discussed control points, but the result is not very good - the points of local extrema do not coincide with the extrema points of the resulting curve.

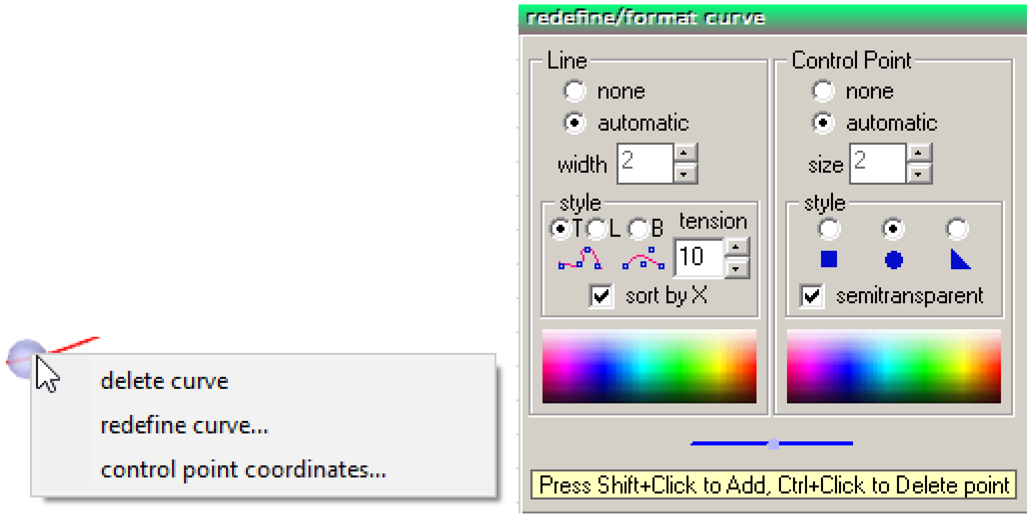
[9] Unfortunately, the well-known CAS systems, for example Maple and Mathematica, are unable to take into account these subjects. They show graph of the derivative of logarithm as hyperbola with two branches. The also incapable to graph isolated points.

Let us check if the control points were set inaccurately. Click the icon of this curve in the legend area. The curve becomes selected (red) (Fig.21 b).
All control points now are visible and located where they should be... Of course, it is all about the mechanism of constructing this curve! It "*does not know*" what we are trying to approximate. Let us fix our curve. Move the mouse pointer to the control point corresponding to the blue point of the local maximum, press the left mouse button and start dragging the selected control point to a more suitable position so that the curve at the point of local maximum really depicts the desired behavior (Fig.21 c). Release the button. ...Done.



a) b) c)
Figure 21

Similarly, we construct and fix the second branch. In this case, pay special attention to the blue marks on the *x*-axis - the inflection points. Do not forget about the point of the graph on the *y*-axis.
To refine the behavior of the curve, one can add few more control points. It is also easy to delete unnecessary control points.
For a more detailed featuring of the curve, place the mouse pointer at a certain control point and press the right mouse button. In the pop-up menu that appears (Fig.22 a), we make a suitable choice: (1) delete the curve, (2) redefine it, or (3) show and/or change the exact coordinates of this control point. Selecting the second option opens a dialog "Redefine/Format curve" (Fig.22 b), which allows to define curve’s color and width, and some characteristics if the control points. Special attention should be paid to the possibility to set the curve style. It can be a T-spline (default), Lagrange curve or B-spline. The name of a Lagrange curve shown in Legend as f_i. One can use it in expressions in the same way as any other explicit function of one variable, for example, $y = f_i(x\text{-}3)+2$. The T- and B-splines names coded in the Legend as h_i and not suitable for this purpose.
The checkbox “*sort by x*” by default is checked to guarantee the impossibility of any two points of a curve to have coincident projections on the *x*-axis.



a) b)

Figure 22

If one suspects a function periodicity, it is enough to sketch a curve at a proper interval, locate the mouse pointer at curve's name in Legend, press the right mouse button and select the option "*replicate along the X-axis*".

If function is odd or even it will be sufficient to sketch the graph only for $x \geq 0$ or $x \leq 0$, press the right mouse button and select the option "*duplicate by central symmetry with regard to the origin*" or "*duplicate by axial symmetry with regard to the Y-axis*".

To check the result of graph sketching one just unhide the graph of $f_1(x)$ and compare it with the constructed sketch.

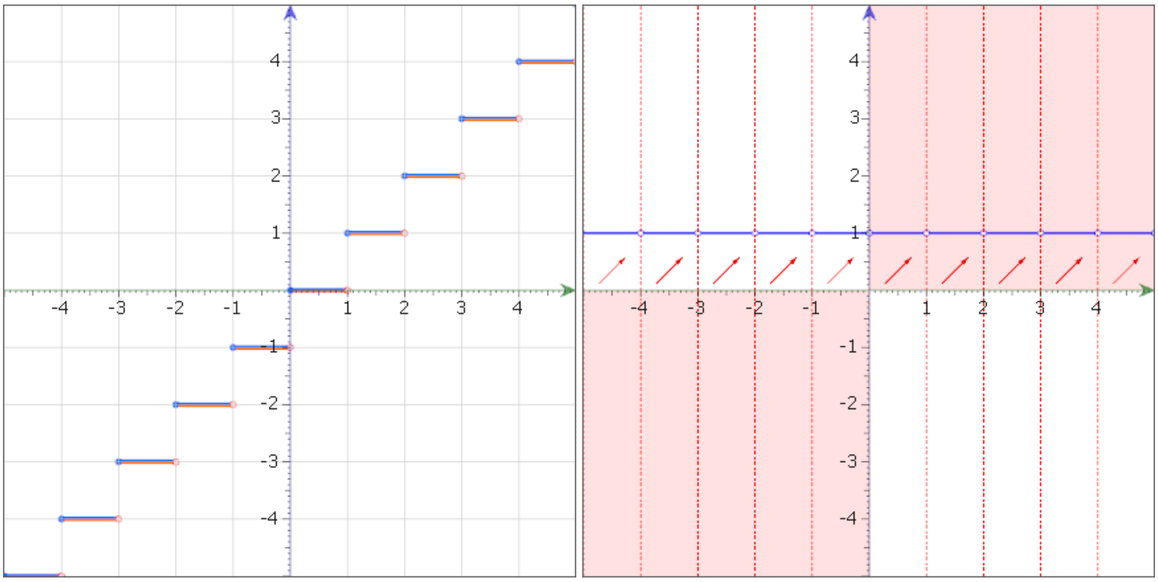
Models of *VisuMatica* are universal. To build a sketch of a graph of another function, it is enough to redefine $f_1(x)$ and the whole show (except of old sketch) rebuilds automatically.

Note, that it is not always advisable to enable all options of the built-in automatic featuring mechanism (Fig.18). Thus, for example, in the case of $f_1(x) = [x]$, the only choice of displaying the local minimum/maximum immediately "gives out" the graph itself[10] (Fig.23 a).

On the other hand, after "slight" change of the function's expression to $f_1(x) = [x] + x$, even enabling the show of all these features is insufficient to grasp the graph's behavior (Fig.23 b).

Displaying the graph of derivative (the blue horizontal line with punctured points of integer abscissas) provides additional information. In a result of analyzing the fact that $f_1(x)' = 1$ on each interval of continuity, we come to the conclusion that this is not enough to sketch the graph of the function. It is necessary to have at least one point in each of these intervals.

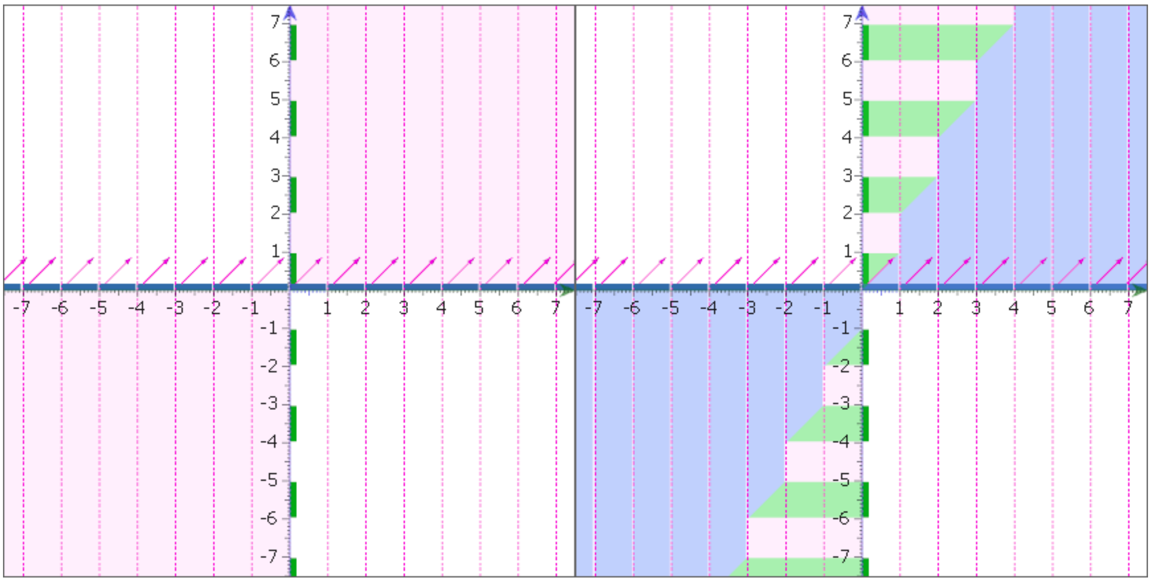
[10] The points, colored in both blue and red, are simultaneously a local maximum and minimum.



a) b)

Figure 23

Displaying the function's Domain (blue) and Range (green) also does not help (Fig.24 a). Only after enabling their show with correspondent trapezoids, we receive an even too expressive help (Fig.24 b). This help remains redundant with only one group of trapezoids.



a) b)

Figure 24

Conclusions

In a result of sketching various examples using similar models, students find that it is not necessary to utilize all the features of the software. One should find the minimal way that provides enough information to solve the problem.

After such materialized activities with the gradual removal of the hint, students are ready to move on to manual sketching.

Computer-free graph sketching skills remain an important part of mathematical culture. In doing so, one can follow, for example, the steps outlined at the beginning of this section, and perform some features of the model by hand. The practice of the choice of the minimum and sufficient set of these steps develops students' intuition and a deep understanding of the qualitative component of the formed knowledge.

Supplementary Electronic Materials

Videos with animations: https://sites.google.com/view/animationssketching/home
VisuMatica in a configuration that supports the above modeling is under construction.

References:

[1] Nodelman V. Counterexamples in Mathematics Education: Why, Where, and How? – Software aspect. Classical subjects. In: The Electronic Journal of Mathematics and Technology, Vol. 12, No. 3 (Oct. 2018), 353-371.

[2] Kunwar R. Functions and Graphs: Basic Techniques of Sketching the Graph and Some Real Life Applications. In: International Journal of Research in Humanities, Arts and Literature. Vol. 7, Issue 5, May 2019, 21-42.

[3] Leinhardt G., Zaslavsky O.,Stein M.K. Functions, Graphs, and Graphing: Tasks, Learning, and Teaching. In: Review of Educational Research. Vol. 60, No. 1 (Spring, 1990), 1-64